Piketty's chain of argument in Le capital au XXI^e siècle

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1. Introduction

Piketty´s book, Le Capital au XXI^e siècle (2013), is fascinating, speculative and provocative, and the literary style of his presentation makes the book an entertaining reading. Piketty persuades the reader by beautiful prose and numerous interesting historical facts; he expects from the reader to believe his bold claims. Yet a reader who wants to go beyond historical description, who looks for a theoretical and conceptual understanding, has sometimes a hard time. Some of my efforts in reading Piketty are presented here.

My interest in and my opinion on the book is best described by quoting from G. Mankiw´s "Yes, r > g. So what" (2015): "Although I admire Piketty and his book, I am not persuaded by his main conclusions. A chain is only as strong as its weakest links, and several links in Piketty´s chain of argument are especially fragile."

For example, I am not at all persuaded by the claimed relevance of Piketty's "fundamental inequality r > g" (see section 2) and the explanatory power of the " fundamental law of capitalism $\beta = \frac{s}{g}$ " (see section 3).[1]

[1] I refer to Piketty for a definition and detailed conceptual discussion of all macro-economic aggregates (theoretical constructs) that are used by Piketty in describing the evolution of a market economy

I shall show that Piketty's "fundamental inequality" - i.e., the national rate r of return on capital exceeds "significantly" the growth rate g of the economy- has no unambiguous implication on the evolution of the capital/income ratio β , which is at the center of Piketty's analysis. In fact, whether the capital/income ratio β is increasing or decreasing does not solely depend on the gap $r-g$, but also on other characteristics of the economy. What matters for an increase in the capital/income ratio β is not a "significantly" positive gap $r-g>0$ (which Piketty considers as the "force de divergence fondamentale"), but a "significantly" high return/growth ratio $\frac{r}{g}$, in the sense that $\frac{r}{g}>\frac{\alpha}{s}$, where α denotes the share of capital income in national income and s the national saving rate (see Proposition 1). Note, according to the historical database the ratio $\frac{\alpha}{s}$ is definitely greater than one in all periods and all countries.

Contrary to Piketty's often repeated claim the "fundamental law of capitalism" - which is nothing else than a simple mathematical limit-theorem - can not "explain" the U-shape of the observed evolution of the capital/income ratio (Piketty, Figure I.2.)

It turns out that the "fundamental law" can be dropped without loss. In fact, only the representation of $\beta_{\tau+n}$ -the capital/income ratio n periods after time τ - as a function of β_τ and the saving- and growth rates from time τ to time $\tau+n$ (formula (9)) is needed to "explain" the U-shape of the evolution of (β_t).

Piketty's main interest in the first two parts of his book is the capital-labor split and the evolution over time of the share α_t of capital income in national income. According to Piketty: "The most fruitful way to understand these changes is to analyze the evolution of the capital/income ratio (that is, the ratio of the total stock of capital to the annual flow of income) rather than focus exclusively on the capital-labor split (that is, the share of income going to capital and labor, respectively). In the past, scholars have mainly studied the latter, largely owing to the lack of adequate data to do anything else." p.42 (77)[2] and "At this stage I am simply explaining the dynamics of the capital/income ratio (a question that can be studied, at least initially, independently

with private property: K_t (market value of the stock of capital at time t), Y_t (flow of national income in period t), g_t (growth rate of the economy), s_t(national saving rate), r_t (national rate of return on capital), β_t (capital/income ratio) etc.

[2]All quotations in English are from the translation, Harvard University Press 2014. In brackets I give the page from the original french publication, Édition du Seuil, 2013.

of the question of how wealth is distributed)." p.170 (268) .

Piketty justifies his two-step approach by referring to the identity $\alpha_t = r_t \cdot \beta_t$ [3]. Thus, if one can establish the circumstances in a market economy where the capital/income ratio β_t is increasing in time (see last part of section 3), then also α_t is increasing, provided one can argue (see Piketty, p.221 (350) and 233 (369)) that the rate of return on capital r_t, if decreasing, changes sufficiently slowly, in the sense that the growth rate of β_t plus the growth rate of r_t is positive.

I will not justify the delicate transition from β to α via r. Here one needs a plausible story (theory) of how the national rate r of return on capital is determined. Thus, my comments on Piketty's book in this note only concern the evolution of the capital/ income ratio β. Section 2 deals with the short-run change and section 3 with the median- and long-run evolution.In the conclusion, section 4, I will comment on the fragile link from β to α.

2. On Piketty's fundamental inequality "$r > g$"

"When the rate of return on capital significantly exceeds the growth rate of the economy (a situation which Piketty denotes symbolically by "$r > g$") than it logically follows that inherited wealth grows faster than output and income" , Piketty, p.26 (55).

Piketty considers this bold claim as fundamental. Indeed, in the subsection " La force de divergence fondamentale" $r > g$" of the Introduction he writes "The fundamental inequality ("$r > g$") will play a crucial role in this book. In a sense, it sums up the overall logic of my conclusions", p.25 (55).

In my opinion, the formulation of Piketty's claim needs a clarification, since - as one can easily show - there exists no assumption on the rate r of return on capital and the growth rate g of the economy alone, that is, an assumption not involving other characteristics of the economy, which implies the conclusion of Piketty's claim.

[3]Unfortunately, Piketty calls the identity, $\alpha_t = r_t \cdot \beta_t$ " la première loi fondamentale du capitalisme". The french word word 'loi' should here and later be understood and be translated by 'relationship' or 'identity'. In the french language 'loi' is often used in this general sense. For example, 'Loi de Walras', in general equilibrium theory, which also is just an aggregated budget identity. To quote an authority, "Les loi, dans la signification la plus étuendue, sont les rapports nécessaires qui dérivent de la nature des choses" Montesquieu, De l'Espirit des Lois.

Piketty asserts that his assumption, "r significantly exceeds g", logically[4] implies his conclusion. Hence, there must exist a formal proof, which requires, however, a clear definition of all assumptions, in particular the logo "$r > g$", and of the conclusion. The crucial point here is, how one describes or models the change over time of the value of capital. The conclusion may be understood in the short-run or in the long-run.

Let me first discuss the short-run interpretation of Piketty's conclusion; that is, the short-run percentage change of the value of capital K is larger than the short-run percentage change of national income Y, i.e.,

$$(1) \qquad \frac{K_{t+1}-K_t}{K_t} > \frac{Y_{t+1}-Y_t}{Y_t} =: g_t$$

One easily shows that inequality (1) is equivalent with inequality

$$(2) \qquad \beta_{t+1} > \beta_t$$

since the capital/income ratio β is defined by $\frac{K}{Y}$.

Certainly, Piketty's fundamental inequality "$r > g$" can not imply inequality (1), since the change $K_{t+1} - K_t$ in the value of capital will not only depend on the rate of return r, the growth rate g and savings and investments in period t, but also on possible price changes of the assets from time t to time $t+1$ and possible capital destruction by war or other crisis.

The change $K_{t+1} - K_t$ of the value of capital from time t to time $t+1$ is usually described in the literature by the following *conceptual decomposition*:

$$(3) \qquad K_{t+1} - K_t = S_t + SP_t + Z_t$$

where S_t denotes the net (of depreciation) national saving flow from time t to time $t+1$, SP_t denotes the surplus (gain or loss) from time t to time $t+1$ of the stock of material capital (assets) at time $t+1$, which is due to price changes of the assets, and Z_t summarizes the market value of all (!) capital changes from time t to time $t+1$ that are not due to saving and investment or price changes during time t to time $t+1$. The conceptual decomposition (3) holds tautologically in every period by definition of its terms.

[4] Actually, he writes sometimes "implique mécaniquement" p.(55) or simply "implique" p.(942) or "implique d' un strict point de vue logique", p.(558), which is translated as "for strictly mathematical reasons", p.351.

According to Piketty, the value of capital as well as the net national savings can be quantified (approximately measured) using the historical database on the structure of national income and national wealth. However, for the surplus component SP_t and, in particular, for the remaining component Z_t one has, at best, some partial and only qualitative knowledge. In the historical database, $SP_t + Z_t$ is treated as a residual $K_{t+1} - K_t - S_t$. Therefore, if one wants to proove Piketty's conclusion, one must restrict the conclusion to the change in the value of capital that is only due to savings and investments alone. Thus, in the following we do not consider the growth rate of capital $\frac{K_{t+1}-K_t}{K_t} = \frac{S_t+SP_t+Z_t}{K_t}$, but only a component of the growth rate, i.e., the savings-investment/capital ratio $\frac{S_t}{K_t}$.

Proposition 1: The component of the growth rate of capital that is due to savings and investment, i.e., $\frac{S_t}{K_t}$*, is greater than the growth rate* g_t *of the economy if and only if*

(4) $\beta_t < \frac{s_t}{g_t}$

or, equivalently if and only if

(5) $r_t > \frac{\alpha_t}{s_t} \cdot g_t$

Indeed, $\frac{S_t}{K_t} = \frac{S_t}{Y_t} \cdot \frac{Y_t}{K_t} = \frac{s_t}{\beta} = r_t \cdot \frac{s_t}{\alpha_t}$, which implies the proposition.

If one defines the saving component of β_{t+1} by $\tilde{\beta}_{t+1} := \frac{K_t+S_t}{Y_{t+1}}$, then the growth rate of capital that is only due to savings and investments is greater than the growth rate g_t of the economy, i.e. $\frac{s_t}{k_t} > g_t$ if and only if $\tilde{\beta}_{t+1} > \beta_t$.

Piketty's fundamental inequality, i.e., "r_t significantly exceeds g_t", has no unambiguous implication; it does not necessarily imply $\beta_{t+1} > \beta_t$. Inequality (5), $\frac{r}{g} > \frac{\alpha}{s}$, may or may not hold, depending on the ratio $\frac{\alpha_t}{s_t}$. It is the ratio $\frac{r}{g}$ that plays the crucial rôle, not the gap $r - g$,which Piketty claims to be "la force de divergence fondamentale".

Example: According to Piketty, $r = 4\%$ significantly exceeds $g = 1\%$. If $\alpha = \frac{1}{3}$ and $s = 5\%$ than inequality (5) does not hold, however, of $\alpha = \frac{1}{4}$ and $s = 10\%$ then (5) is satisfied.

In principle, the ratio $\frac{\alpha}{s}$ can take any positive value, however, in the available historical database in every period and for all countries, the ratio $\frac{\alpha_t}{s_t}$ is always greater than one. Will there ever exist a "happy" economy where the share of capital income in national income is smaller than the national saving rate s ?

Conclusion: We have derived the following clarification of Piketty's claim, which was quoted at the beginning of this section:

"When the ratio $\frac{r_t}{g_t}$ of the rate of return on capital and the growth rate of the economy significantly exceeds one - in the sense that $\frac{r_t}{g_t} > \frac{\alpha_t}{s_t}$ - then it logically follows that the component of the growth rate of capital that is due to savings is larger than the growth rate of output and income" or in compact notation: *the inequality $\frac{r_t}{g_t} > \frac{\alpha_t}{s_t}$ is equivalent to the inequality $\tilde{\beta}_{t+1} > \beta_t$.*

Remark: Piketty's fundamental inequality "$r > g$ " for an economy á la Marx: the population consists of two (disjoint) social classes; capitalists, who own capital and have capital income, yet do not work and laborers, who own no capital, yet work and have labor income. The historical data, which Piketty wants to "explain", are not realizations of such economies á la Marx. Thus, we are arguing in the following in a fictitious model-economy!

Let $Y^K$and S^Kdenote total capital income and total saving of the subpopulation of capitalists, analogously $Y^L$and S^L for the subpopulation of laborers. Then the national saving rate s of the economy is $s = \frac{S^K+S^L}{Y} = \frac{S^K}{Y^K} \cdot \frac{Y^K}{Y} + \frac{S^L}{Y^L} \cdot \frac{Y^L}{Y}$.

Since $\frac{Y^K}{Y} = \alpha$ and $Y = Y^K + Y^L$ we obtain

(6) $$s = s^K \cdot \alpha + s^L \cdot (1-\alpha)$$

where $s^K = \frac{S^K}{Y^K}$ and $s^L = \frac{S^L}{Y^L}$ denote the saving rate of the subpopulation of capitalists and laborers, respectively.

If there are many more (depending on α) laborers than capitalists (implying a very unequal distribution of capital across the population) then capital income $Y^K = \alpha Y$ is split among relatively few capitalists while labor income $Y^L = (1-\alpha)Y$ is split among many laborers. Consequently, the relatively few capitalists will have a high individual income, while the many laborers will have a low individual income, near to the existence level if there are sufficiently many laborers. Thus, s^K will be quite large, possibly near to one, and s^Lwill be small or even zero. In the latter case, where $s^L = 0$, the national saving rate s is equal to αs^K, and hence the inequality (5), i.e., $\frac{s}{\alpha} r > g$, reduces to $s^K \cdot r > g$, which may be interpreted as "r sufficiently exceeds g". I conjecture that Piketty had such a Marxian economy with sufficient inequality of the capital distribution in mind when he formulated his fundamental inequality "$r > g$".

We remark that the decomposition of the national saving rate

$$s = \alpha s^K + (1-\alpha)s^L$$

can also be defined for non-Marxian economies, provided one defines $s^K$and s^Las the national saving rate out of capital income and out of labor income, respectively. These concepts are defined on the assumption that individual saving S^i depends on individual income y^i (labor plus capital income) and S^i is split into saving out of capital income and out of labor income by $S^{iK} = \frac{y^{iK}}{y^i} \cdot S^i$ and $S^{iL} = \frac{y^{iL}}{y^i} \cdot S^i$. The inequality (5), i.e., $\frac{s}{\alpha} \cdot r > g$, then becomes $(s^K - s^L + \frac{s^L}{\alpha}) \cdot r > g$. But this inequality only approaches Piketty's "$r > g$" if the saving rate s^L out of labor income is very small.

3. The evolution of the capital/income ratio β

The conceptual decomposition of the change of the value of capital (3), $K_{t+1} - K_t$, in the last section implies an analogous conceptual decomposition for the capital/income ratio β_{t+1}:

$$\beta_{t+1} = \tfrac{1}{1+g_t}\beta_t + \tfrac{s_t}{1+g_t} + \tfrac{SP_t}{Y_{t+1}} + \tfrac{Z_t}{Y_{t+1}} \tag{7}$$

The component $\tilde{\beta}_{t,1} = \frac{1}{1+g_t}\beta_t + \frac{s_t}{1+g_t} =: L_t(\beta_t)$ is called the *saving component of* β_{t+1} and the residual $\beta_{t+1} - \tilde{\beta}_{t+1}$ is denoted by $\frac{R_t}{Y_{t+1}}$.

Starting at time τ with the capital/income ratio β_τ, one obtains by (7)

$$\beta_{\tau+1} = L_\tau(\beta_\tau) + \tfrac{R_\tau}{Y_{\tau+1}}$$

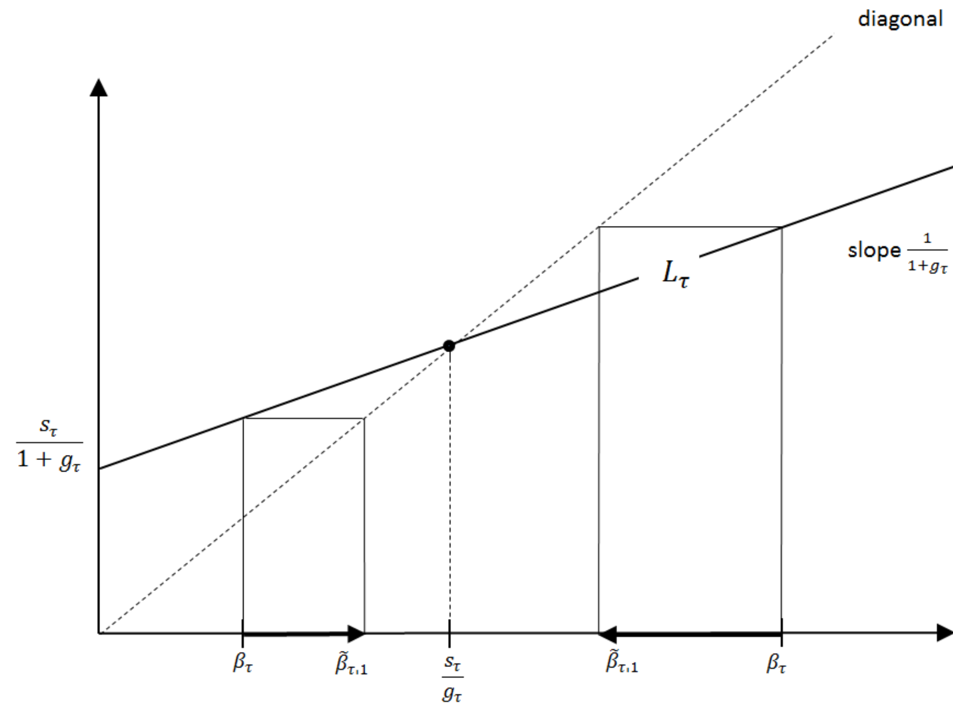

Figure 1: every $\beta_\tau$moves toward $\frac{s_\tau}{g_\tau}$; $\tilde{\beta}_{t,1} > \beta_\tau$ if $\beta_\tau < \frac{s_\tau}{g_\tau}$ and $\tilde{\beta}_{t,1} < \beta_\tau$ if $\beta_\tau > \frac{s_\tau}{g_\tau}$.

One period later, applying (7) for $t = \tau + 1$ yields

$$\begin{aligned}\beta_{\tau+2} &= L_{\tau+1}(\beta_{\tau+1}) + \frac{R_{\tau+1}}{Y_{\tau+2}} \\ &= L_{\tau+1}(L_\tau(\beta_\tau)) + \frac{R_\tau}{(1+g_t)Y_{\tau+1}} + \frac{R_{\tau+1}}{Y_{\tau+2}} \\ &= L_{\tau+1} \cdot L_\tau(\beta_\tau) + \frac{R_\tau + R_{\tau+1}}{Y_{\tau+2}}.\end{aligned}$$

Finally, n periods later, one obtains

$$\beta_{\tau+n} = L_{\tau+n-1} \cdot L_{\tau+n} \cdot ... \cdot L_\tau(\beta_\tau) + \frac{R_\tau + ... + R_{\tau+n-1}}{Y_{\tau+n}} \tag{8}$$

The n-fold iteration $L_{\tau+n-1} \cdot ... \cdot L_\tau(\beta_\tau)$ is denoted by $\tilde{\beta}_{\tau,n}$ and is called *the saving component of the capital/income ratio* $\beta_{\tau+n}$; $\tilde{\beta}_{\tau,n}$ can be computed from the data: the starting capital/income ratio β_τ and the saving and growth rates from time τ to time $\tau+$

$n-1$ [5]. For the remaining component of the capital/income ratio $\beta_{\tau,n}$, i.e., $\frac{R_\tau+...+R_{\tau+n-1}}{Y_{\tau+n}}$, one has only partial and qualitative information. Therefore, it is an interesting and relevant question to ask whether the saving component $\tilde{\beta}_{\tau,n}$ alone explains an essential part of the capital/income ratio $\beta_{\tau+n}$. For small n, say in the short run n=1, one does not expect a positive answer, since the surplus SP_τ or the capital destruction Z_τ might be larger than savings. To answer the question in the median-run, say for n=50, one needs historical data for the horizon from time τ to time $\tau+50$, and "let the data speak" (Piketty, p.175 (277)) "...if one compares the level of private wealth in 2010 predicted by the savings flows observed between 1970 and 2010 (to the year with the initial wealth observed in 1970) with the actual observed levels of wealth in 2010, one finds that the two numbers are quite similar (une grande proximité) for most countries". In our notation, Piketty has shown that β_{2010} *and* $\tilde{\beta}_{1970,40}$ *are quite similar* ($\tilde{\beta}_{1970,40}$ explains 70-80% of β_{2010}) *for eight countries*: U.S., Japan, Germany, France, U.K., Italy, Canada and Australia.

In a later publication, Piketty and Zuchmann (2014) compute the saving component $\beta_{1870,140}$ for U.S., U.K., Germany and France. For this much longer horizon the computed saving component $\beta_{1870,140}$ explains "surprisingly well the observed capital/income ratios of the year 2010". Piketty explains this surprising empirical observation by the fact that the surplus term SP_τ *can be positive or negative* and therefore, there is a possibility that they cancel out, at least partially. Together with economic growth it might well happen that the term $\frac{SP_\tau+...+SP_{\tau+n-1}}{Y_{\tau+n}}$ is small. If the event of war or other crises are relatively seldom (transient) then, again with economic growth even the term $\frac{Z_\tau+...+Z_{\tau+n-1}}{Y_{\tau+n}}$ might be small.

Remark: Piketty leaves open whether - based on his empirical findings - he implicitly assumes that the remainder term can always be neglected in long-run arguments, i.e., $\beta_{\tau+n}$ - $\tilde{\beta}_{\tau,n}$ is small for large n. In any case, Piketty does not distinguish in his notation the capital/income ratio $\beta_{\tau+n}$ in period $\tau+n$ from the saving component $\tilde{\beta}_{\tau,n}$.

Motivated by Piketty's interesting empirical findings, we analyze in the following the median or long-run behavior of the savings component $\tilde{\beta}_{\tau,n}$ in more detail.

[5] "These two macrosocial parameters themselves depend on millions of individual decisions influenced by any number of social, economic, cultural, psychological, and demographic factors and may vary considerably from period to period and country to country. Furthermore, they are largely independent of each other.", p.199 (315)

A straightforward calculation yields

$$\tilde{\beta}_{\tau,n} = [\prod_{t=\tau}^{\tau+n-1} \frac{1}{1+g_t}] \cdot \beta_\tau + \sum_{t=\tau}^{\tau+n-1} [\prod_{v=t}^{\tau+n-1} \frac{1}{1+g_v}] \cdot s_t \tag{9}$$

Thus, the evolution of the saving component $(\tilde{\beta}_{\tau+n})_{n=1,2,...}$, starting at time τ with β_τ, is completely determined by the evolution of the national saving rates and the growth rates of the economy.

Piketty does not state this at first glance complex looking expression (9) for $\tilde{\beta}_{\tau+n}$ in his book, since he wants to be simple. In his drive for simplicity he considers the hypothetical case where the saving rate s_t and the growth rate g_t of the economy *are constant* from time τ to time $\tau + n$.

Then the expression (9) simplifies to

$$\tilde{\beta}_{\tau,n} = \begin{cases} (\frac{1}{1+g})^n \beta_\tau + \frac{s}{g}(1 - (\frac{1}{1+g})^n) & \text{if } g \neq 0 \\ \beta_\tau + ns & \text{if } g = 0 \end{cases}$$

Thus, for sufficiently large n, the ratio $\frac{s}{g}$ is a good approximation for$\tilde{\beta}_{\tau+n}$. In the unlimited long-run, when $g_t = g$ and $s_t = s$ forever, one obtains

$$\lim_{n\to\infty} \tilde{\beta}_{\tau,n} = \begin{cases} \frac{s}{g} & \text{if } g \neq 0 \\ \infty & \text{if } g = 0 \text{ and } s > 0 \end{cases} \tag{10}$$

In this hypothetical case (which is counterfactual at least for large n; see Piketty's historical database), the evolution of the saving component $\tilde{\beta}_{\tau,n}$ can easily be visualized graphically, since the straight lime L_τ in Figure 1 is the same in every step of the iteration.

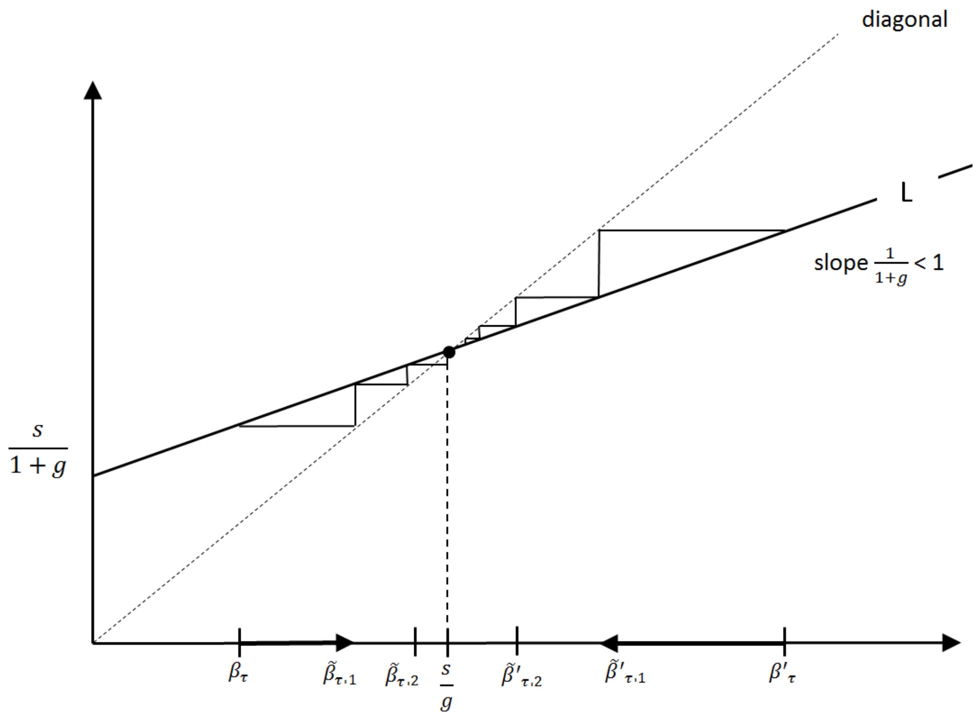

Figure 2: $\tilde{\beta}_{\tau,n}$ converges monotonically increasing (decreasing) to $\frac{s}{g}$ if $\beta_\tau < \frac{s}{g}$ ($\beta_\tau > \frac{s}{g}$).

This simple and purely mathematical limit result (10) is baptized by Piketty "loi fundamentale du capitalisme", p.166 (262), which he denotes symbolically by "$\beta = \frac{s}{g}$", even though one can only prove $\lim_{n\to\infty} \tilde{\beta}_{\tau+n} = \frac{s}{g}$ and not $\lim_{n\to\infty} \beta_{\tau+n} = \frac{s}{g}$ (see the above Remark and the discussion on p. 169 (267) and footnote 5(1) on p. 170 (269). Even more surprisingly is Piketty's claim that the limit result can explain, at least partially, the observed U-shape of the evolution of the capital/income ratio. "If we look at the whole period 1910-2010 or 1870-2010, we find that the global evolution of the capital/income ratio is very well explained by the dynamic law $\beta = \frac{s}{g}$ ", p. 187 (295). Certainly, to be simple is desirable, but here I believe Piketty goes too far. For example, consider the horizon from 1970 to 2010. Piketty shows in Figure 4.3, 4.4 and 5.3 that the capital/income ratio from 1970 to 2010 has a strong increasing pattern for all countries. Then he writes, p. 173 (273): "This structural evolution is explained by three sets of factors, which complement and reinforce one another... The most important factor in the long run is slower growth (the ralentissement de la croissance)... together with a high rate of saving (avec le maintien d'une épargne élevée), automatically *gives rise* to a structural increase in the long-run capital/income ratio, *owing to the law* $\beta = \frac{s}{g}$

(au travers de)". One wonders how the "fundamental law", i.e., the asymptotic result $\lim_{n\to\infty}\tilde{\beta}_{\tau,n} = \frac{s}{g}$, can logically imply that the *decreasing* growth rate from 1970 to 2010, together with a *maintained* level of the rate of savings leads to a definite increase of the saving component of the capital/income ratio from a low level in 1970 to the high level in 2010? The "fundamental law" is based on stationary evolutions of the saving- and growth rates. In the hypothetical case, when the high growth rate 1970 would remain forever, the saving component would then converge to $\frac{s}{g_{1970}}$. This may be compared to the hypothetical case where the low growth rate of 2010 would remain forever. The saving component then converges to $\frac{s}{g_{2010}}$, which is larger than $\frac{s}{g_{1970}}$. Why does this comparison of two hypothetical cases imply what one wants to explain? In my opinion, Piketty's "fundamental law of capitalism" is not needed at all. Indeed, the desired explanation follows directly - without a limit argument - from the definition or from the formula (9) of the saving component $\tilde{\beta}_{\tau+n}$ as the following Proposition 2 shows:

Proposition 2: The evolution of the saving component $(\tilde{\beta}_{\tau+n})_{n=0,1,...}$

(a) *is increasing in n, if* $\beta_\tau < \frac{s_\tau}{g_\tau}$ *and if* $\frac{s_{\tau+n}}{g_{\tau+n}}$ *is increasing in n, e.g. the growth rate is decreasing and the saving rate remains at the same level.*

(b) *is decreasing in n, if* $\beta_\tau > \frac{s_\tau}{g_\tau}$ *and if* $\frac{s_{\tau+n}}{g_{\tau+n}}$ *is decreasing in n, e.g. the saving rate is decreasing and the growth rate remains at the same level.*

The historical data satisfies the assumptions of Proposition 2 (a); $\beta_{1970} < \frac{s_{1970}}{g_{1970}}$ (β_{1970} is between 2 and 3 depending on the country and $\frac{s_{1970}}{g_{1970}}$ is between 5 and 6) and the ratio $\frac{s_\tau}{g_\tau}$ is increasing from 1970 to 2010. The strong increase of the evolution of the measured capital/income ratios in Figure 5.3 are, of course, not only due to the increase of the saving components, but also to the remainder terms, see Piketty p. 173 (273).

The sharp decrease of the capital/income ratio from 1913 to 1950 (see Piketty, Figures 4.4 and 4.5) is, to a large extent, due to economic policy decisions, wars and other crises (Piketty, p. 148 (234), yet also to a decrease in the saving component, which can be explained by Proposition 2 (b). Indeed, β_{1913} is large (between 6 and 7) and the saving rate s_t is very sharply decreasing, leading to a decreasing ratio $\frac{s_t}{g_t}$ from 1913 to 1950.

Proof of Proposition 2: By assumptions, $\beta_\tau < \frac{s_\tau}{g_\tau}$, which implies (see Figure 1) , that $\beta_\tau < \tilde{\beta}_{\tau,1} < \frac{s_\tau}{g_\tau}$. By assumption $\frac{s_\tau}{g_\tau} < \frac{s_{\tau+1}}{g_{\tau+1}}$, hence $\tilde{\beta}_{\tau,1} < \frac{s_{\tau+1}}{g_{\tau+1}}$. Therefore in the next step of iteration, i.e. $\tilde{\beta}_{\tau,2} = L_{\tau+1}(\tilde{\beta}_{\tau,1})$, one obtains $\tilde{\beta}_{\tau,1} < \tilde{\beta}_{\tau,2} < \frac{s_{\tau+1}}{g_{\tau+1}}$. Again by assumption $\frac{s_{\tau+1}}{g_{\tau+1}} < \frac{s_{\tau+2}}{g_{\tau+2}}$ and hence, $\tilde{\beta}_{\tau,2} < \frac{s_{\tau+2}}{g_{\tau+2}}$. And so on.

In summary: In modifying somewhat Piketty's arguments, we derived an "explanation scheme" for the observed U-shape of the evolution of the capital/income ratio for all countries where sufficient data are available. Also the qualitative difference in the U-shape of the various countries can be explained. The explanation scheme is based on the conceptual decomposition (7). The analysis is *purely descriptive*; we explained an observed pattern of Piketty's historical database. The "explanation" is certainly not causal; there might exist other explanations.

Can one use this explanation scheme for looking into the future? Even if one accepts the principle 'what worked in the past will work in the future', one can not forecast(predict) the future evolution of the capital/income ratio since this requires forecasts (predictions) of the growth and saving rates for which one has no theory, because the growth and saving rates are *endogenous* aggregate characteristics of the *unknown real world process*. However, one may ask a *purely hypothetical (mathematical) question: Under which hypothetical or fictitious properties of the evolution of the growth rate g_t and the saving rate s_t is the long-run evolution of the saving component $\tilde{\beta}_{\tau,n}$, as defined by* (9),

(i) *converging to a limit*

(ii) *remaining bounded or*

(iii) *increasing without bound?*

ad (i): *convergence*

If the (non-negative) saving- and growth rates are convergent, $s_t \to s$ and $g_t \to g$ with $g > 0$, then for every starting point β_τ at time τ the sequence $(\tilde{\beta}_{\tau+n})_{n=1,2,\ldots}$ is convergent and the limit does not depend on β_τ: $\lim_{n\to\infty} \tilde{\beta}_{\tau,n} = \frac{s}{g}$.

ad (ii): *boundedness*

If $g_t \geq g > 0$ for all $t \geq \tau$, then the sequence $(\tilde{\beta}_{\tau,n})_{n=1,2,\ldots}$ is *bounded*. Furthermore, if g_t varies in the interval $[\underline{g}, \overline{g}]$ with $\underline{g} > 0$ and $s_t \epsilon [\underline{s}, \overline{s}]$ for all $t \geq \tau$ then the bounded sequence $(\tilde{\beta}_{\tau,n})_{n=1,2,\ldots}$ need not be convergent, yet every limit-point of the sequence belongs to the interval $[\frac{\underline{s}}{\overline{g}}, \frac{\overline{s}}{\underline{g}},]$. Finally, if the positive growth rate g_t *converges to zero,* then the sequence $(\tilde{\beta}_{\tau,n})_{n=1,2,...}$ *is still bounded* provided the saving rates s_t *converge sufficiently fast to zero,* i.e., the partial sum $\sum_{t=\tau}^{\tau+n-1} s_t$ of the saving rates from τ to $\tau+n$ does not tend to infinity for increasing n.

ad (iii): *unboundedness*

One knows by (ii) that the sequence $(\tilde{\beta}_{\tau+n})_{n=1,2,...}$ can only tend to infinity if the growth rate tends to zero. In this case, either national income remains bounded (i.e., g_t converges fast to zero; $\lim_{n\to\infty}\prod_{t=\tau}^{\tau+n-1}\frac{1}{1+g_t} > 0$, e.g., if the growth rate is zero from some future time on) or national income tends to infinity (i.e., g_t converges slowly to zero; $\lim_{n\to\infty}\prod_{t=\tau}^{\tau+n-1}\frac{1}{1+g_t} = 0$). In the first case the sequence $(\tilde{\beta}_{\tau,n})_{n=1,2,...}$ is *unbounded* if the saving rate does not converge fast to zero. In the second case $(\tilde{\beta}_{\tau,n})_{n=1,2,...}$is *unbounded* if the partial sum $\sum_{t=\tau}^{\tau+n-1} s_t$ of the national saving rates from period τ to period $\tau+n$ grows faster to infinity than national income $Y_{\tau+n}$. Indeed, since $g_t \geq 0$, it follows that $\prod_{t=v,v>\tau}^{\tau+n-1}\frac{1}{1+g_t} \geq \prod_{t=\tau}^{\tau+n-1}\frac{1}{1+g_t}$ and therefore $(\prod_{t=\tau}^{\tau+n-1}\frac{1}{1+g_t})(\sum_{t=\tau}^{\tau+n-1} s_t) \leq \tilde{\beta}_{\tau+n}$, which implies the claim.

4. Conclusion

Piketty's chain of arguments can be summarized by the following chart:

$\beta_\tau, (g_t, s_t)_{\tau\geq t} \underset{link1}{\Longrightarrow} (\tilde{\beta}_{\tau,n})_{n+1,...} \underset{link2}{\Longrightarrow} (\beta_{\tau+n})_{n=1,...} \underset{link3}{\Longrightarrow} (\alpha_{\tau+n})_{n=1,...} \overset{final}{\underset{link}{\Longrightarrow}}$ individual capital- and labor income distributions across the population.

Of particular interest is whether this chain of arguments justifies Piketty's claim that "the process by which wealth is accumulated and distributed contains powerful forces pushing towards divergence [in the sense that the share α of capital income in national income is increasing] or at any rate towards an extremely high level of inequality", p.27 (56).

The first link, the transition from the capital/income ratio β_τ and the evolution of the growth- and saving rates to the evolution of the saving component, $\tilde{\beta}_{\tau+n}$ is given by formula (9). This link is simple and solid. The evolution of $\tilde{\beta}_{\tau+n}$ is well-understood for various alternative scenarios of (g_t, s_t), e.g., Proposition 2 or the unlimited long-run results when the growth rate tends to zero.

The second link, the transition from the saving component $\tilde{\beta}_{\tau+n}$ to the capital/income ratio $\beta_{\tau+n}$, is given by the conceptual decomposition (8)

$$\beta_{\tau+n} = \tilde{\beta}_{\tau,n} + \frac{R_\tau + ... + R_{\tau+n-1}}{Y_{\tau+n}}$$

Piketty specifies this link in his long-run analysis by *taking for granted* or *assuming as valid* the following two implications:

1) if the saving component $\tilde{\beta}_{\tau,n}$ is convergent, then the capital/income ratio $\beta_{\tau+n}$ is convergent to same limit.

2) if the saving component $\tilde{\beta}_{\tau,n}$ tends to infinity, then also the capital/income ratio $\beta_{\tau+n}$ tends to infinity.

Whether these two implications are valid, depends, of course, on the long-run evolution of the remainder terms. Consequently, Piketty's specification of the second link is fragile, since it is based on an unspecified assumption on the evolution of asset prices and on the frequency of capital destructions.

The third link, the transition from the capital/income ratio β to the share α of capital income in national income, is given by the identity $\alpha = r \cdot \beta$.

The link is only useful if one has a solid and satisfactory explanation of how the national rate of return on capital r is determined. Recall, r is defined by the ratio of national capital income and the value of capital; it is not an average of "individual" rates of return on capital.

Certainly one agrees with Piketty when he writes "the rate of return on capital is determined by the following two forces: first, technology, and second, the abundance of capital stock", p.212 (336). But to combine these two forces by means of the (ill-defined) notion of an "aggregate production function" in capital and labor and the notion of "marginal productivity of capital" in order to obtain the national rate of return on capital is difficult to accept, certainly for those readers who are familiar with the literature on aggregation in production (see, for example, Malinvand (1983) or Felipe and Fisher (2003).

What does one know on the joint evolution of (r_t, β_t)? Are they (stochastically) negatively associated? Let again the historical data speak: "experience suggests that the predictable rise in the capital/income ratio will not necessarily lead to a significant drop in the return on capital... The most likely outcome is thus that the decrease in the rate of return will be smaller than the increase in the capital/income ratio" p. 233 (369). Does this observation support the hypothesis that the sum of the growth rate of the capital/income ratio, $\frac{\beta_{t+1}-\beta_t}{\beta_t}$, and the growth rate of the rate of return, $\frac{r_{t+1}-r_t}{r_t}$, is (likely to be) positive? If this is the case, then one can show that an increasing capital/income ratio β leads to an increasing share α of capital income in national income. More work, empirical and theoretical, is needed to justify the third link.

5. References

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